Graph Minors Theory

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Edge contraction

Definition (edge contraction)

Let e = xy be an edge of a graph G = (V, E). By G/e we denote the graph obtained from G by contracting the edge e into a new vertex v_e , which becomes adjacent to all the former neighbours of x and of y.

Formally G/e is a graph (V', E') with vertex set

$V' := (V \backslash \{x, y\}) \cup \{v_e\}$

(where v_e is the new vertex, i.e. $v_e \notin V \cup E$) and edge set:

$E' := \{vw \in E \mid \{x, y\} \cap \{v, w\} = \emptyset\} \cup$

$\{v_e w \mid xw \in E \backslash \{e\} \text{ or } yw \in E \backslash \{e\}\}$.

Minor

Definition (minor)

A graph H is a minor of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges[1].

Definition (another minor definition)

Any graph H that can be produced from G by successive application of these reductions is called a minor of G:
(a) delete an edge,
(b) contract an edge,
(c) delete an isolated node.

Topological minor

Definition (topological minor)

A graph H is called a topological minor of a graph G if a subdivision of H is isomorphic to a subgraph of G.

Theorem

every topological minor of a graph is also its (ordinary) minor.

Theorem

every minor with maximum degree at most 3 of a graph is also its topological minor.

Interesting works and results

During the past decades graphs minors theory has been developed so well that one can name the series of papers by Robertson and Seymour(Graph Minors I...XXIII) as the most important work in graph theory. many interesting results and approaches are present, just to name a few we have the following short list:

- algorithmic aspects: finding $O(n^3)$ algorithm for solving k-Disjoint path problem for fixed k.
- algorithmic aspects: if we bound tree-width of instances of many NP-hard problems we can solve those in poly-time.
- Wagner conjecture:For every minor-closed family of graphs the set of forbidden minors is finite.
- probabilistic methods: Hajos conjecture disproof and recent works in proving it for large girth.

Hadwiger conjecture

One of the most important and challenging open problems in graph theory is Hadwiger's conjecture:

Conjecture (Hadwiger 1943)

for every integer $r > 0$ and every graph G:
$\chi(G) \geq r \;\Rightarrow\; G \succ K_r$

key facts:

- this conjecture is true for $r < 7$ and still open for greater values.
- as $\chi(K_{t,t}) = 2 \;\wedge\; K_{t,t} \succ K_t$ nothing can be said conversely.

Hajos conjecture

The Hajos conjecture is a strengthened version of Hadwiger conjecture which states:

Conjecture (Hajos)

for every integer $r > 0$ and every graph G:
$\chi(G) \geq r \;\Rightarrow\; G \succ_t K_r$

Key facts about this conjecture:

- Hajos conjecture has been failed in general.
- conjecture is true for $r \leq 4$ and false for $r \geq 7$ and cases 5 & 6 are still open.
- Erdos has showed with probabilistic methods that almost every graph which is large enough is a counter example for this conjecture.

Wagner conjecture

Wagner conjecture or as it is known today (Robertson Seymour theorem) is one of the most important works in graph theory in past decades. the theorem states:

Theorem

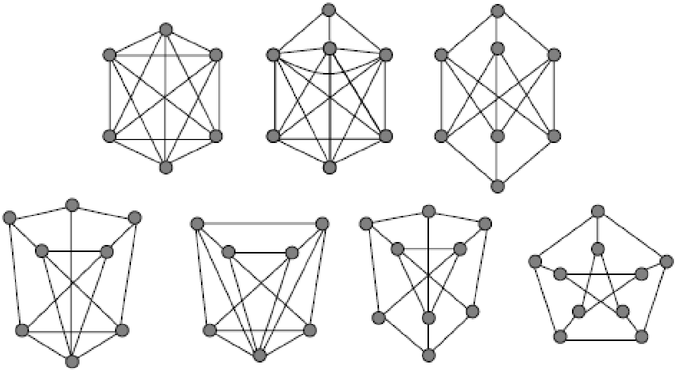
For every minor-closed family of graphs the set of forbidden minors is finite.

this theorem generalizes the planar graphs theorem in which we have K_5 & $K_{3,3}$ as forbidden minors.
a variation of this theorem is for being linklessly embeddable:

Theorem

A graph is linklessly embeddable if and only if it does not contain any of the seven graphs in Petersen family as a minor.

Wagner conjecture



The Petersen family

Wagner conjecture

the following theorem is equivalent to this theorem which states the set of all finite graphs with minor relation is well quasi ordered

Theorem

For every infinite sequence $G_1, G_2, \ldots$ of graphs, there exist distinct integers $i < j$ such that G_i is a minor of G_j .

the proof of Wagner's conjecture is one the main result from series of 23 papers named Graph Minors I to Graph Minors XXIII published by Robertson and Seymour from 80s to 2004. these works are considered as one the most important projects in graph theory.

Mader works

Mader has proved that for every graph H there is a constant C_H such that every graph G not containing H as a minor satisfies $|E(G)| \leq C_H|V(G)|$, but determining the best possible constant C_H for a given graph H is a question that has been answered for very few graphs H.
In fact Mader has shown that:

Theorem (Mader 1967)

There is a function $h : \mathbb{N} \to \mathbb{N}$ such that every graph with average degree at least h(r) contains K_r as a topological minor for every $r \in \mathbb{N}$.

The function obtained in this theorem is $h(r) = 2^{r(r-1)/2}$.
as we will see in the following sections this bound is so loose.

Bounds for K_r (topological)minor free graphs

Two bounds for K_r (topological)minor free graphs have been found and these bounds are sharp up to a constant c as a function of r.

Theorem (Bollobas & Thomason and independently Komlos & Szemeredi)

There exists a $c \in \mathbb{R}$ such that, for every $r \in \mathbb{N}$, every graph G of average degree $d(G) > cr^2$ contains K_r as a topological minor.

Theorem (Kostochka 1982; Thomason 1984[2])

There exists a $c \in \mathbb{R}$ such that, for every $r \in \mathbb{N}$, every graph G of average degree $d(G) > cr\sqrt{\log r}$ contains K_r as a minor.

Minor and Girth

There are many results in minors theory which uses girth of a graph just to name a few we have these two theorems. the interesting fact is the second theorem shows that only big enough girth can force certain minors with high minimum degree.

Theorem (Mader 1997)

For every graph H of maximum degree $d \geq 3$ there exists an integer k such that every graph G of minimum degree at least d and girth at least k contains H as a topological minor.

Theorem (Thomassen 1983)

Given an integer k, every graph G with girth $g(G) \geq 4k - 3$ and $\delta(G) \geq 3$ has a minor H with $\delta(H) \geq k$.

$K_{1,t}$ minor free graphs

It is easy to see that every n-vertex graph with more than $\frac{1}{2}(t-1)n$ edges contains $K_{1,t}$ as a minor (indeed, as a subgraph), and if t divides n then there is an n-vertex graph with exactly $\frac{1}{2}(t-1)n$ edges with no $K_{1,t}$ minor (the disjoint union of n/t copies of K_t).
The extremal example for above statement is not connected and the answer when we restrict ourselves to connected graphs is different.

Theorem (G. Ding, P. Seymour and T. Johnson, 2001)

Let $t \geq 3$ and $n \geq t+2$ be integers. If G is an n-vertex connected graph with no $K_{1,t}$ minor, then $|E(G)| \leq n + \frac{1}{2}t(t-3)$ and for all n, t this is best possible.

$K_{s,t}$ minor free graphs

More generally, what is the maximum number of edges in graphs with no $K_{s,t}$ minor when $s > 1$? If we take a graph each component of which is a clique of size t, and add $s-1$ more vertices each adjacent to all others, then the resulting n-vertex graph has no $K_{s,t}$ minor, and the number of edges is : $(t+2s+3)(n-s+1)/2+(s-1)(s-2)/2$.

Key facts:

- Kostochka and Prince have a proof of this for all sufficiently large t.
- it is open for $s = 4, 5$.
- for $s \geq 6$ Kostochka and Prince have counterexamples.
- Kostochka and Prince proved the following:

$K_{s,t}$ minor free graphs

Theorem (A. Kostochka and N. Prince[3])

Let s,t be positive integers with $t > (240s\log_2 s)^{8s\log_2 s+1}$. Then every graph with average degree at least $t+3s$ has a $K_{s,t}$ minor, and there are graphs with average degree at least $t + 3s - 5\sqrt{s}$ that do not have a $K_{s,t}$ minor.

Theorem (A. Kostochka[4])

let s and t be positive integers such that

$$t > t_0(s) := \max 4^{15s^2+2}, (240s\log_2(s))^{8s\log_2 s+1} \tag{1}$$

then every (s+t)-chromatic graph has a $k^*_{s,t}$-minor

$K_{2,t}$ minor free graphs

Theorem (Chudnovsky, Reed and Seymour 2011[5])

Let $t \geq 2$, and let G be a graph with $n > 0$ vertices and with no $K_{2,t}$ minor. Then

$$|E(G)| \leq \frac{1}{2}(t+1)(n-1)$$

.

key facts:

- Myers had previously proved the theorem for all $t \geq 10^{29}$.
- the bound is best possible when n - 1 is a multiple of t.
- for $n = \frac{3}{2}t$ the bound is about $\frac{1}{2}tn$ but the best known result is $\frac{5}{12}tn$.

$K_{2,t}$ minor free graphs sketch of proof

Fix $t \geq 2$ and suppose the theorem is false for that value of t. So there is a minimal counterexample, that is, a graph G with the following properties:

- G has no $K_{2,t}$ minor.
- $|E(G)| > \frac{1}{2}(t + 1)(|V(G)| - 1)$.
- $|E(G')| > \frac{1}{2}(t + 1)(|V(G')| - 1)$ for every graph G' with no $K_{2,t}$ minor and $|V(G')| < |V(G)|$.

Since $|E(G)| > \frac{1}{2}(t + 1)(n - 1)$ it follows that $n \geq t + 2$.

$K_{2,t}$ minor free graphs sketch of proof

The proof is technical and so long but the main idea is outlined here:

- It is first shown that G is 5-connected, and so $t \geq 6$.
- Then following lemma is prooved which is main argument of the proof:
  Let $W \subseteq V(G)$ be connected with $|W| \geq 2$. If $t \geq 11$ then $|N(W)| \geq t + 3$.
- Then small t cases are handeled and an edge is found with large neighbourhood.

Probabilistic methods and Hajos conjecture

Using the chernoff bound we have the following result:

Theorem (Erdos, Fajtlowicz)

For n sufficiently large there exists graphs with chromatic number at least $\frac{n}{2\log_2 n}$ and no topological minor of $K_{8\sqrt{n}}$.

- majority of large graphs are counterexamples.
- we should have $n \geq 2^{30}$
- Hajos conjecture is true for large girth.

Theorem (Kuhn, Osthus 2006)

Let $r \geq 1$ be a natural number. Every graph of minimum degree at least r and girth at least 27 contains a subdivision of K_{r+1} .

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