Kneser's Conjecture and its Generalizations

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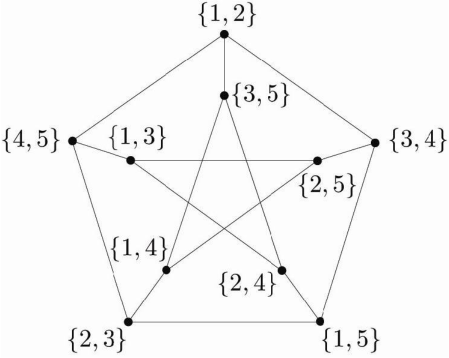

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- A subset S of $[m]$ is called 2-stable if $x, y \in S$ & $x \neq y$, then $2 \leq |x - y| \leq m - 2$. Schrijver graph $SCH(m,n)$ is the subgraph of $\mathrm{KG}(m,n)$ induced by all 2-stable n-subsets.

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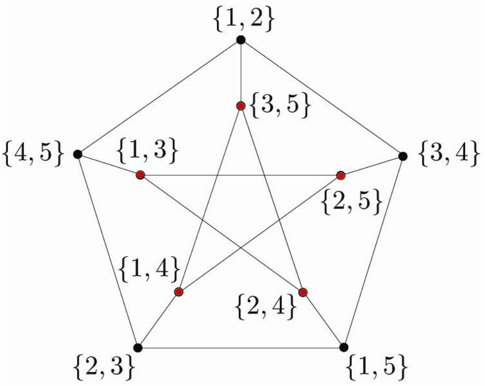

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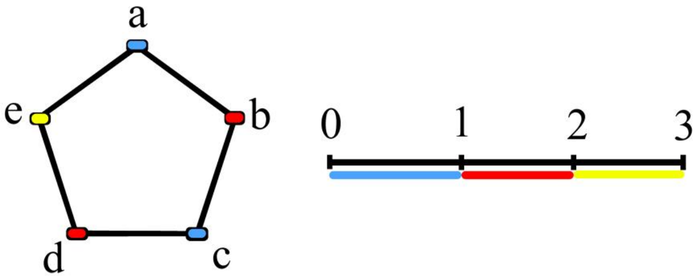

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$$C_i = \{A \in V(\mathrm{KG}(m,n)) | A \cap \{1,2,\ldots,i\} = \{i\}\}, 1 \leq i \leq m-2n$$

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- (Lovász, 1978) For all $m \geq 2n$, $\chi(\mathrm{KG}(m,n)) = m-2n+2$.
- (Schrijver) For all $m \geq 2n$, $\chi(SCH(m,n)) = m-2n+2$.

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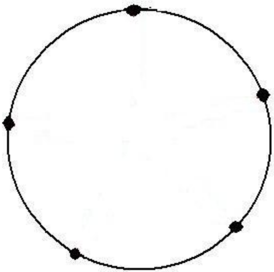

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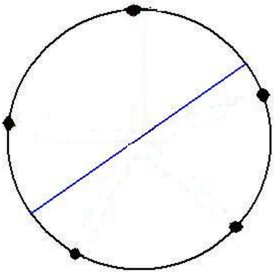

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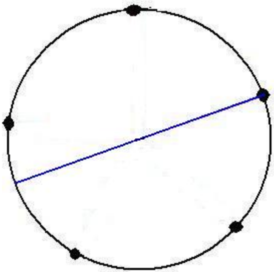

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- There is $1 \leq i \leq d+1$ and $x \in S^d$ with $x, -x \in A_i$. This leads to a contradiction: We have two disjoint k-tuples of color i, one in $H(x)$ and one in $H(-x)$.

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- Consider the Schrijver graph $\mathrm{SG}(n,k)$ and set $d := n - 2k$.
- There exists a $(2k+d)$-point set $X \subseteq S^d$ such that under a suitable identification of X with $[n]$, every open hemisphere contains a stable k-tuple.
- For contradiction, suppose that a proper $(d+1)$-coloring of $\mathrm{KG}(n,k)$ has been chosen.
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- Proof: By induction on the number of closed sets in the cover of S^d.

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Dolnikov's Theorem

- Let $\mathcal{H} = (X, \mathcal{F})$ be a hypergraph. $\mathcal{H}$ is called 2-colorable if there exists a map $c : X \to \{\textbf{Green}, \textbf{Red}\}$ such that no edge is monochromatic.

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- Let the 2-colorability defect, denoted by $\mathrm{cd}_2(\mathcal{H})$, be the minimum size of a subset $Y \subseteq X$ such that the system of the sets of $\mathcal{F}$ that contain no points of Y is 2-colorable. In symbols,

$$\mathrm{cd}_2(\mathcal{H}) = \min\{|Y| : (X \backslash Y, \{F \in \mathcal{F} : F \cap Y = \emptyset\}) \text{ is } 2\text{−colorable}\}.$$

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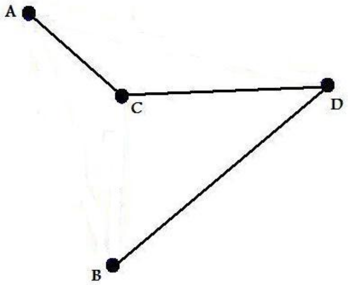

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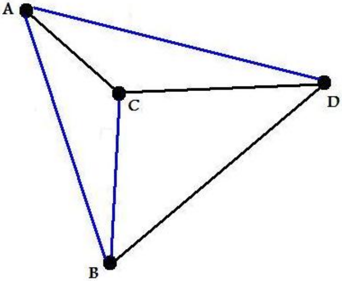

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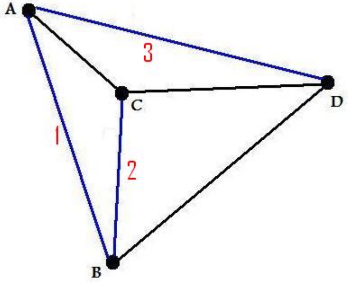

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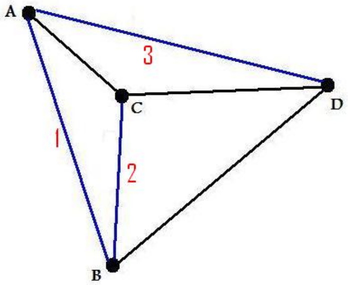

$$A \longrightarrow \{1,3\} \quad B \longrightarrow \{1,2\}$$
$$C \longrightarrow \{2\} \quad D \longrightarrow \{3\}$$

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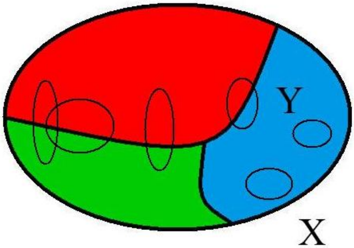

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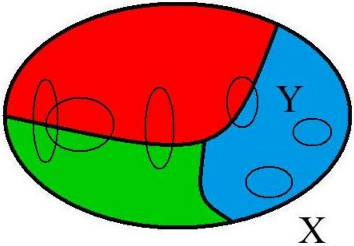

- (Dolnikov's Theorem) For any finite set system $(X, \mathcal{F})$, we have $\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}_2(\mathcal{F})$.

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- Let $d := \chi(\mathrm{KG}(\mathcal{F}))$. As in the above proof of the Lovász-Kneser theorem, we identify the ground set of X with a point set $X \subset S^d$ in general position (no $d+1$ points on an equator).

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- Let Y be the point in X lying in the equator separating the two hemispheres. Color the point of X in $H(x)$ red, those in $H(-x)$ blue. Therefore, $(X \setminus Y, \{F \in \mathcal{F} : F \cap Y = \emptyset\})$ is 2-colorable. Since $|Y| \leq d$, $\mathrm{cd}_2(\mathcal{F}) \leq \chi(KG(\mathcal{F})) = d$.

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Thank You!