# Introduction to Algebraic Topology

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*Tehran, Iran*

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# Topological space

- A topological space is a pair $(X, \mathcal{O})$, where $X$ is a (typically infinite) ground set and $\mathcal{O}$ is a set system, whose members are called the open sets, such that . $\varnothing \in \mathcal{O}$ and $X \in \mathcal{O}$, the intersection of finitely many open sets is an open set, and so is the union of an arbitrary collection of open sets.

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- A homeomorphism of topological spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is a bijection $\varphi : X_1 \longrightarrow X_2$ such that for every $U \subset X_1$, $\phi(U) \in O_2$ if and only if $U \in O_1$ . In other words, a bijection is a homeomorphism if and only if both $\phi, \phi^{-1}$ are continuous.

# Deformation Retract

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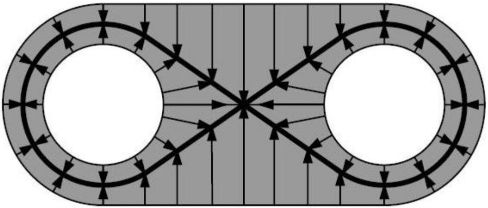

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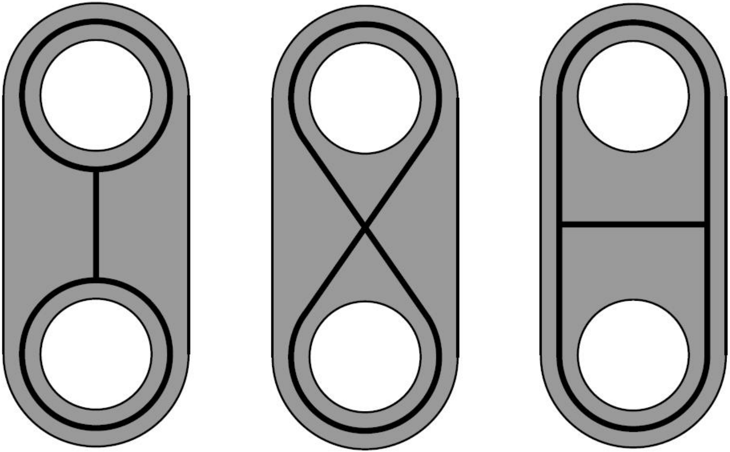

# Homotopy Equivalent

- Two continuous maps $f, g : X \longrightarrow Y$ are homotopic (written $f \sim g$) if there is a continuous interpolation between them; that is, a family $\{f_t\}_{t\in[0,1]}$ of maps $f_t : X \longrightarrow Y$ depending continuously on $t$ (i.e., the associated bivariate mapping $F(x,t) := f_t(x)$ is a continuous map $X \times [0,1] \longrightarrow Y$ , similar to deformation retraction above) such that $f_0 = f$ and $f_1 = g$.

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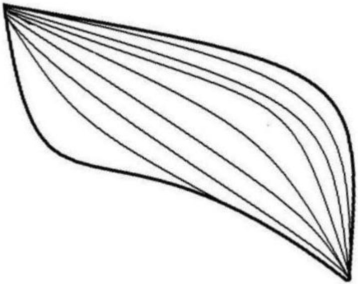

# Homotopy Equivalent

- Two spaces $X$ and $Y$ are homotopy equivalent (or have the same homotopy type) if there are continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the composition $f \circ g : Y \longrightarrow Y$ is homotopic to the identity map $id_Y$ and $g \circ f \sim id_X$.

# Simplicial Complexes

- Let $v_0, v_1, \ldots, v_k$ be points in $\mathbb{R}^d$. They called affinely dependent if there are real numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$, not all of them $0$, such that $\sum_{i=0}^{k} \alpha_i v_i = 0$ and $\sum_{i=0}^{k} \alpha_i = 0$. Otherwise, $v_0, v_1, \ldots, v_k$ are called affinely independent.

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- The points $v_0, v_1, \ldots, v_k$ are affinely independent if and only if $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent.

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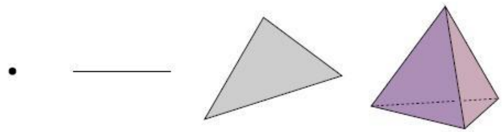
- **Examples of Simplices**- Here are examples of simplices: a point, a line segment, a triangle, and a tetrahedron: These examples have dimensions 0, 1, 2, and 3, respectively.



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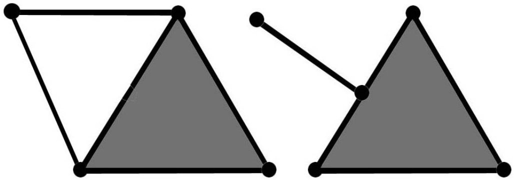

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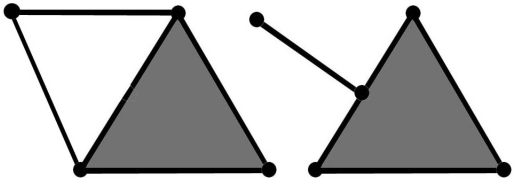

- The union of all simplices in $\Delta$ is the polyhedron of $\Delta$ and denoted by $\| \Delta \|$.

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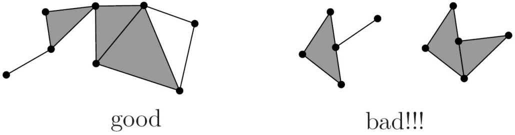

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- A subcomplex of a simplicial complex $\Delta$ is a subset of $\Delta$ that is itself a simplicial complex (that is, it is closed under taking faces).
- An important example of a subcomplex is the k-skeleton of a simplicial complex $\Delta$. It consists of all simplices of $\Delta$ of dimension at most k, and we denote it by $\Delta^{\leq k}$

# Triangulation

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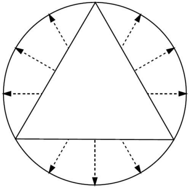
- The simplest triangulation of the sphere $S^{n-1}$ is the boundary of an n-simplex, that is, the subcomplex of $\sigma$ obtained by deleting the single n-dimensional simplex (but retaining all of its proper faces). Indeed, the boundary of an n-simplex is homeomorphic to $S^{n-1}$, as can be seen using the central projection:



# Abstract Simplicial Complexes

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- **Example** The collection of sets $K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is an abstract simplicial complex. To obtain the simplex $\Delta^{\{1,2,3\}}$ one would need to add to this collection the set $\{1,2,3\}$.
- In this situation we call $\Delta$ a geometric realization of $K$.

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- Each geometric simplicial complex $\Delta$ determines an abstract simplicial complex. The points of the abstract simplicial complex are all vertices of the simplices of $\Delta$, so we set $V := V(\Delta)$, and the sets in the abstract simplicial complex are just the vertex sets of the simplices of $\Delta$

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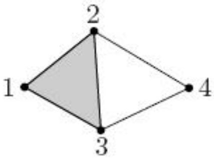
- For example, for the geometric simplicial complex

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$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}\}$.

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- A simplicial collapse of $\Delta$ is the removal of all simplices $\gamma$ such that $\tau \subseteq \gamma \subseteq \sigma$.

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- When $\Delta_1$ and $\Delta_2$ are two simplicial complexes such that there exists a sequence of collapses leading from $\Delta_1$ to $\Delta_2$, this is shown by the notation $\Delta_1 \searrow \Delta_2$.
- **Theorem.** A sequence of collapses yields a strong deformation retraction, in particular, a homotopy equivalence.

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- **Definition**. Let K and L be two abstract simplicial complexes. A simplicial mapping of $K$ into $L$ is a mapping $f : V(K) \to V(L)$ that maps simplices to simplices, i.e., such that $f(F) \in L$ whenever $F \in K$.

# Simplicial Mappings

- For every simplicial mapping $f : V(\Delta_1) \to V(\Delta_2)$ , it induces a natural continuous function $||f|| : ||\Delta_1|| \to ||\Delta_2||$ by setting:

$$||f|| : \sum_{v_i \in V(\Delta_1)} t_i v_i \longmapsto \sum_{v_i \in V(\Delta_1)} t_i f(v_i)$$

This is easy to check that this map is a continuous map between spaces. If $f$ is injective, then $||f|| : \Delta_1 \to \Delta_2$ is injective too, and if f is an isomorphism, then $||f||$ is a homeomorphism.

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- Simplicial complexes are **connection between Combinatorics and Topology**.
- We summarize this connection :

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3. Conversely, if a topological space admits a triangulation, it can be described purely combinatorially by an abstract simplicial complex. (This description is not unique)
4. A continuous map, even between triangulated spaces, generally cannot be described by a simplicial map, but it is true under suitable conditions:

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2. These kinds of theorems called simplicial approximation theorem.
3. In fact there exists a large variety of complexes whose description is purely combinatorial. In the following slides we survey different situations in which complexes defined by combinatorial data arise

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- For a graph $G$ and a subset $S \subset V(G)$ of its vertices, we let $G[S]$ denote the corresponding induced graph.
- **Definition.** Given an arbitrary graph $G$, we let $Cl(G)$ denote the abstract simplicial complex whose set of vertices is $V(G)$ and whose simplices are all subsets $S \subset V(G)$ such that $G[S]$ is a complete graph. This complex is also called the clique complex of the graph $G$.

# Independence Complexes

- Given a graph $G$, its complement $\overline{G}$ is the graph with the same set of vertices such that $(v, w)$ is an edge of $\overline{G}$ if and only if $v \neq w$ and $(v, w)$ is not an edge of $G$. A set of vertices $S \subset V(G)$ is called independent if for all $v, w \in S$ we have $(v, w)$ is not in $E(G)$.

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- **Definition.** For an arbitrary graph $G$, the independence complex of $G$, called Ind($G$), is the abstract simplicial complex whose set of vertices is $V(G)$ and whose simplices are all the independent sets (anticliques) of $G$.
- Since independent sets of G are the same as the cliques of $\overline{G}$, we see that Ind($G$) is isomorphic to $Cl(\overline{G})$ as an abstract simplicial complex.

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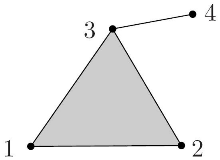
- The face poset of a simplicial complex $K$ is the poset $P(K)$, which is the set of all nonempty simplices of $K$ ordered by inclusion.

# Order Complex

- For example, the simplicial complex



# Order Complex

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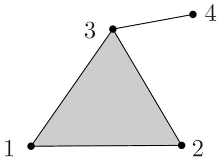
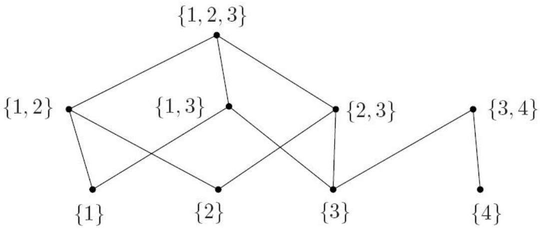



- has the face poset



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- Let $n \in \mathbb{N}$. The partition lattice $\Pi_n$ is the partially ordered set whose elements are all set partitions of the set $[n]$, and the partial order is that of refinement. The partition lattice has a minimal element $\{1\}\{2\}\ldots\{n\}$ and a maximal element $[n]$. See the figure below. It is a theorem which asserts that $\Delta(\Pi_n)$ is homotopy equivalent to a wedge of $(n-1)!$ copies of $S^{n-2}$.

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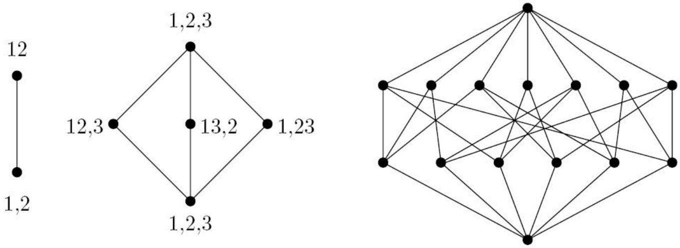
- 

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- **Example.** For a group $G$ and a prime $p$, we denote by $S_p(G)$ the poset of all nontrivial p-subgroups of $G$, i.e., the subgroups whose cardinality is a power of p. Furthermore, we let $Ap(G)$ denote the poset of all nontrivial elementary abelian p-subgroups of $G$.

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- It turns out that the order simplicial complex $\Delta(A_p(G))$ is homotopy equivalent to $\Delta(S_p(G))$. This has led to several investigations of these and other posets of various families of subgroups of a given group G.
- It is interesting to understand the connections between the group-theoretic properties of the considered families of subgroups and the topological properties of the order complexes of the corresponding posets.

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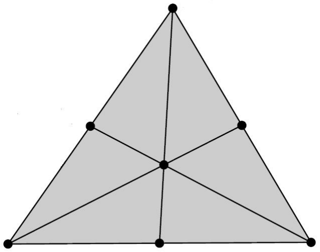

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# Neighborhood Complex

- Let $G$ be a graph. The neighborhood complex of $G$ is the abstract simplicial complex $\mathcal{N}(G)$ defined as follows: its vertices are all non-isolated vertices of $G$, and its simplices are all the subsets of $V(G)$ that have a common neighbor.

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- Let $N(v)$ denote the set of neighbors of $v$, i.e.,

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- Furthermore, for an arbitrary subset $A \subset V(G)$, we let $N(A)$ denote the set of common neighbors of $A$, i.e. $N(A) = \bigcap_{v \in A} N(v)$.

# Lov´asz Complex

- The definition of $N$ gives an order-reversing map $N : \mathcal{F}(\mathcal{N}(G)) \to \mathcal{F}(\mathcal{N}(G))$. It can be seen that $N^3 = N$ and that $N^2(A) \supseteq A$, for any $A \subset V(G)$.

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- The Lov´asz complex plays an important role in the proof of the so-called Kneser conjecture which is an old problem in graph theory.
- In fact Lov´asz gives a very nontrivial translation of the expression " the chromatic number of the graph $G$ is not equal to $k$" in the realm of graph theory to the expression "the simplicial complex $\mathcal{L}o(G)$ is $k$-connected" in the realm of (algebraic) topology.

Thank You!