Combinatorial Generalization of The Borsuk-Ulam Theorem

Hossein Hajiabolhassan

Department of Mathematical Sciences

Shahid Beheshti University, G.C.
Tehran, Iran

Workshop On Topological Combinatorics
Shahid Beheshti University, G.C.
Tehran, Iran

Wednesday, October 21, 2009

Triangulation

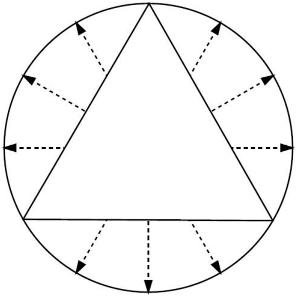
- Let X be a topological space. A simplicial complex Δ such that $X \cong \Delta$ (X is homeomorphic to Δ) is called a triangulation of X.
- The boundary of an n-simplex is homeomorphic to S^{n-1}, as can be seen using the central projection:

Triangulation

- Let X be a topological space. A simplicial complex Δ such that $X \cong \Delta$ (X is homeomorphic to Δ) is called a triangulation of X.
- The boundary of an n-simplex is homeomorphic to S^{n-1}, as can be seen using the central projection:

Triangulation

- Let X be a topological space. A simplicial complex Δ such that $X \cong \Delta$ (X is homeomorphic to Δ) is called a triangulation of X.
- The boundary of an n-simplex is homeomorphic to S^{n-1}, as can be seen using the central projection:



Cross Polytope

- Write $e_1, e_2, \ldots, e_n$ for the vectors of the standard orthonormal basis of R^n (e_i has a 1 at position i and 0's elsewhere). Define a simplicial complex C_{n-1} (cross polytope) as follows:
- The vertex set of T is equal to $\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$.
- A subset $F \subseteq \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ forms the vertex set of a proper face of the cross polytope if and only if there is no $i \in [n]$ with both $e_i \in F$ and $-e_i \in F$.

Cross Polytope

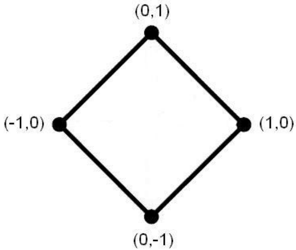
- Write $e_1, e_2, \dots, e_n$ for the vectors of the standard orthonormal basis of R^n (e_i has a 1 at position i and 0's elsewhere). Define a simplicial complex C_{n-1} (cross polytope) as follows:
- The vertex set of T is equal to $\{\pm e_1, \pm e_2, \dots, \pm e_n\}$.
- A subset $F \subseteq \{\pm e_1, \pm e_2, \dots, \pm e_n\}$ forms the vertex set of a proper face of the cross polytope if and only if there is no $i \in [n]$ with both $e_i \in F$ and $-e_i \in F$.

Cross Polytope

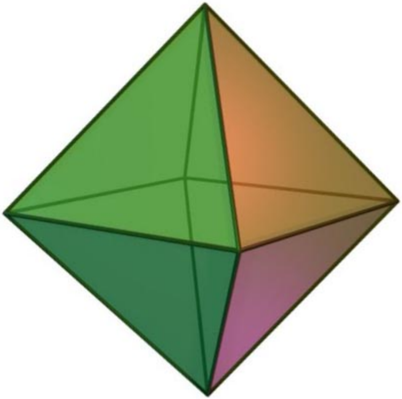
- Write $e_1, e_2, \ldots, e_n$ for the vectors of the standard orthonormal basis of R^n (e_i has a 1 at position i and 0's elsewhere). Define a simplicial complex C_{n-1} (cross polytope) as follows:
- The vertex set of T is equal to $\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$.
- A subset $F \subseteq \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ forms the vertex set of a proper face of the cross polytope if and only if there is no $i \in [n]$ with both $e_i \in F$ and $-e_i \in F$.

Cross Polytope

- Write $e_1, e_2, \ldots, e_n$ for the vectors of the standard orthonormal basis of R^n. Define a simplicial complex C_{n-1} (cross polytope) as follows:
- The vertex set of T is equal to $\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$.
- A subset $F \subseteq \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ forms the vertex set of a proper face of the cross polytope if and only if there is no $i \in [n]$ with both $e_i \in F$ and $-e_i \in F$.



Cross polytopes

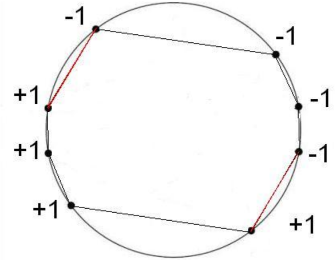


Tucker's Lemma for S^n

- (Tucker's Lemma) Let T be a triangulation of S^n that is antipodally symmetric. Assume that $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v. Then there exists a 1-simplex (an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.

Tucker's Lemma for S^n

- (Tucker's Lemma) Let T be a triangulation of S^n that is antipodally symmetric. Assume that $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v. Then there exists a 1-simplex (an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.



Sketch of Proof

- (Tucker's Lemma) Let T be a triangulation of S^n that is antipodally symmetric. Assume that $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v. Then there exists a 1-simplex (an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.
- If there is no edge which is complementary, then one can check that λ deduce an antipodal map from S^n to S^{n-1} which is a contradiction.
- For any vertex $v \in T$, if $\lambda(v) = +i$ (resp. $\lambda(v) = -i$), then set $f(v) := e_i$ (resp. $f(v) := -e_i$).
- It is easy to check that f is a Z_2-map from T to C_{n-1} (cross polytope of dimension $n-1$) which is a contradiction.

Sketch of Proof

- (Tucker's Lemma) Let T be a triangulation of S^n that is antipodally symmetric. Assume that $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \dots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v. Then there exists a 1-simplex (an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.
- If there is no edge which is complementary, then one can check that λ deduce an antipodal map from S^n to S^{n-1} which is a contradiction.
- For any vertex $v \in T$, if $\lambda(v) = +i$ (resp. $\lambda(v) = -i$), then set $f(v) := e_i$ (resp. $f(v) := -e_i$).
- It is easy to check that f is a Z_2-map from T to C_{n-1} (cross polytope of dimension $n-1$) which is a contradiction.

Sketch of Proof

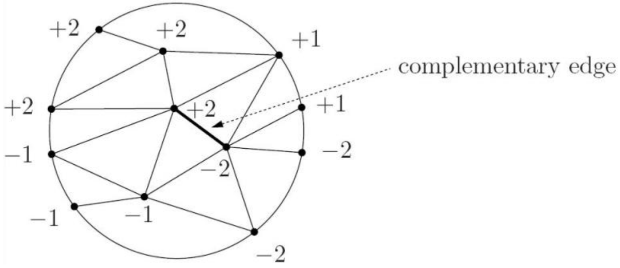
- (Tucker's Lemma) Let T be a triangulation of S^n that is antipodally symmetric. Assume that $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v. Then there exists a 1-simplex (an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.
- If there is no edge which is complementary, then one can check that λ deduce an antipodal map from S^n to S^{n-1} which is a contradiction.
- For any vertex $v \in T$, if $\lambda(v) = +i$ (resp. $\lambda(v) = -i$), then set $f(v) := e_i$ (resp. $f(v) := -e_i$).
- It is easy to check that f is a Z_2-map from T to C_{n-1} (cross polytope of dimension $n-1$) which is a contradiction.

Sketch of Proof

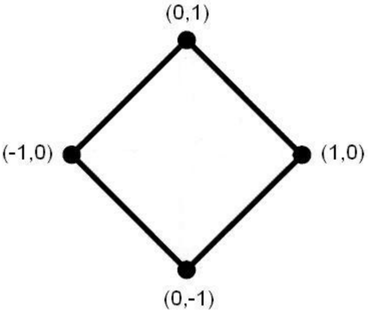
- (Tucker's Lemma) Let T be a triangulation of S^n that is antipodally symmetric. Assume that $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v. Then there exists a 1-simplex (an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.
- If there is no edge which is complementary, then one can check that λ deduce an antipodal map from S^n to S^{n-1} which is a contradiction.
- For any vertex $v \in T$, if $\lambda(v) = +i$ (resp. $\lambda(v) = -i$), then set $f(v) := e_i$ (resp. $f(v) := -e_i$).
- It is easy to check that f is a Z_2-map from T to C_{n-1} (cross polytope of dimension $n-1$) which is a contradiction.

TUCKER'S LEMMA FOR B^n

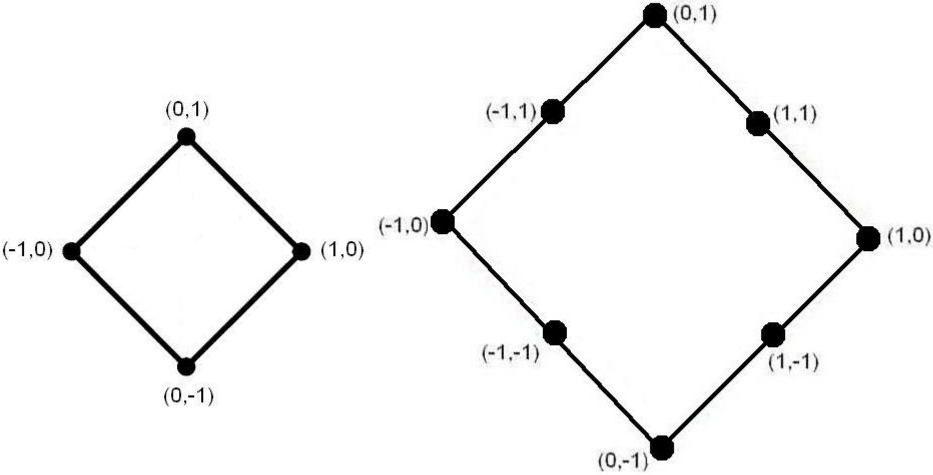
- (Tucker's Lemma) Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$ be a labeling of the vertices of T that satisfies $\lambda(-v) = -\lambda(v)$ for every vertex v on the boundary. Then there exists a 1-simplex(an edge) in T that is complementary; i.e., its two vertices are labeled by opposite numbers.



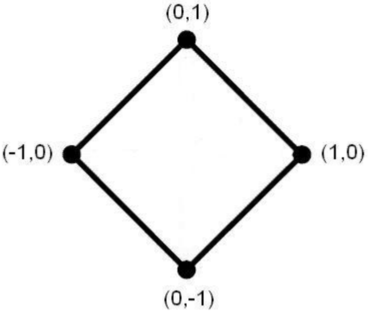
Cross polytopes

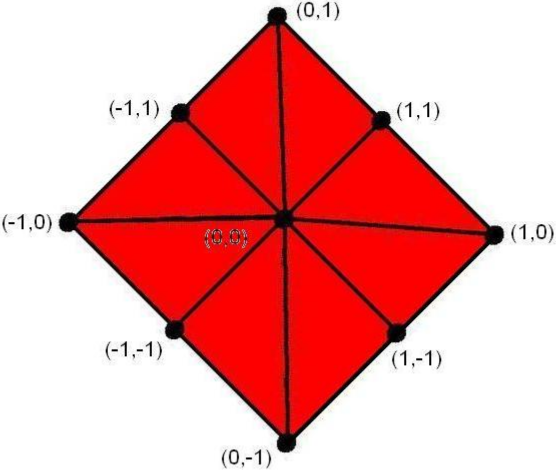


Cross polytopes



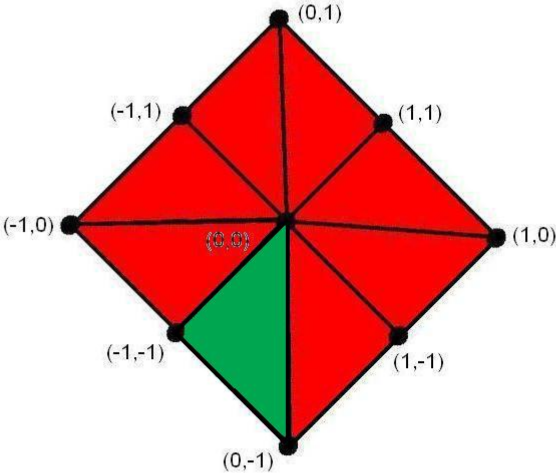
Cross polytopes





Cross polytopes

u and v belong to a simplex if $u \leq v$ ($u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$)



Combinatorial Tucker's Lemma

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Tucker's Lemma) Let

$$\lambda : \{-1, 0, +1\}^n \longrightarrow \{-1, +1, -2, +2, \dots, -n, +n\}$$

be a labeling that satisfies $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$. Then there exist $u, v \in V_n$ such that $u \leq v$ and $\lambda(u) = -\lambda(v)$.

Combinatorial Tucker's Lemma

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Tucker's Lemma) Let

$$\lambda : \{-1, 0, +1\}^n \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$$

be a labeling that satisfies $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$. Then there exist $u, v \in V_n$ such that $u \leq v$ and $\lambda(u) = -\lambda(v)$.

Combinatorial Tucker's Lemma

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Tucker's Lemma) Let

$$\lambda : \{-1, 0, +1\}^n \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$$

be a labeling that satisfies $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$. Then there exist $u, v \in V_n$ such that $u \leq v$ and $\lambda(u) = -\lambda(v)$.

Combinatorial Tucker's Lemma

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\preceq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \preceq v$ if $u_i \preceq v_i$ for any $1 \leq i \leq n$ where $0 \preceq -1$ and $0 \preceq +1$.
- (Combinatorial Tucker's Lemma) Let

$$\lambda : \{-1, 0, +1\}^n \longrightarrow \{-1, +1, -2, +2, \ldots, -n, +n\}$$

be a labeling that satisfies $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$. Then there exist $u, v \in V_n$ such that $u \preceq v$ and $\lambda(u) = -\lambda(v)$.

Lovász-Kneser Theorem

- Set $[n] = \{1, 2, \dots, n\}$.
- Let $w = (w_1, w_2, \dots, w_n) \in V_n$.
- Set $P(w) := \{i \in [n] : w_i = +1\}$.
- Set $N(w) := \{i \in [n] : w_i = -1\}$.
- Consider an arbitrary linear ordering on $2^{[n]}$ that refines the partial ordering according to size, i.e., if $|A| < |B|$ then $A < B$.
- On the contrary, suppose that

$$c : V(\mathrm{KG}(n, k)) \longrightarrow \{2k, \dots, n\}$$

is a proper coloring of $\mathrm{KG}(n, k)$ with $n - 2k + 1$ colors.

Lovász-Kneser Theorem

- Set $[n] = \{1, 2, \ldots, n\}$.
- Let $w = (w_1, w_2, \ldots, w_n) \in V_n$.
- Set $P(w) := \{i \in [n] : w_i = +1\}$.
- Set $N(w) := \{i \in [n] : w_i = -1\}$.
- Consider an arbitrary linear ordering on $2^{[n]}$ that refines the partial ordering according to size, i.e., if $|A| < |B|$ then $A < B$.
- On the contrary, suppose that

$$c : V(\mathrm{KG}(n, k)) \longrightarrow \{2k, \ldots, n\}$$

is a proper coloring of $\mathrm{KG}(n, k)$ with $n - 2k + 1$ colors.

Lovász-Kneser Theorem

- Set $[n] = \{1, 2, \ldots, n\}$.
- Let $w = (w_1, w_2, \ldots, w_n) \in V_n$.
- Set $P(w) := \{i \in [n] : w_i = +1\}$.
- Set $N(w) := \{i \in [n] : w_i = -1\}$.
- Consider an arbitrary linear ordering on $2^{[n]}$ that refines the partial ordering according to size, i.e., if $|A| < |B|$ then $A < B$.
- On the contrary, suppose that

$$c : V(\mathrm{KG}(n, k)) \to \{2k, \ldots, n\}$$

is a proper coloring of $\mathrm{KG}(n, k)$ with $n - 2k + 1$ colors.

Lovász-Kneser Theorem

- Set $[n] = \{1, 2, \ldots, n\}$.
- Let $w = (w_1, w_2, \ldots, w_n) \in V_n$.
- Set $P(w) := \{i \in [n] : w_i = +1\}$.
- Set $N(w) := \{i \in [n] : w_i = -1\}$.
- Consider an arbitrary linear ordering on $2^{[n]}$ that refines the partial ordering according to size, i.e., if $|A| < |B|$ then $A < B$.
- On the contrary, suppose that

$$c : V(\mathrm{KG}(n, k)) \longrightarrow \{2k, \ldots, n\}$$

is a proper coloring of $\mathrm{KG}(n, k)$ with $n - 2k + 1$ colors.

Lovász-Kneser Theorem

- Set $[n] = \{1, 2, \ldots, n\}$.
- Let $w = (w_1, w_2, \ldots, w_n) \in V_n$.
- Set $P(w) := \{i \in [n] : w_i = +1\}$.
- Set $N(w) := \{i \in [n] : w_i = -1\}$.
- Consider an arbitrary linear ordering on $2^{[n]}$ that refines the partial ordering according to size, i.e., if $|A| < |B|$ then $A < B$.
- On the contrary, suppose that

$$c : V(\mathrm{KG}(n, k)) \longrightarrow \{2k, \ldots, n\}$$

is a proper coloring of $\mathrm{KG}(n, k)$ with $n - 2k + 1$ colors.

Lovász-Kneser Theorem

- Set $[n] = \{1, 2, \dots, n\}$.
- Let $w = (w_1, w_2, \dots, w_n) \in V_n$.
- Set $P(w) := \{i \in [n] : w_i = +1\}$.
- Set $N(w) := \{i \in [n] : w_i = -1\}$.
- Consider an arbitrary linear ordering on $2^{[n]}$ that refines the partial ordering according to size, i.e., if $|A| < |B|$ then $A < B$.
- On the contrary, suppose that

$$c : V(\mathrm{KG}(n,k)) \to \{2k, \dots, n\}$$

is a proper coloring of $\mathrm{KG}(n,k)$ with $n - 2k + 1$ colors.

Lovász-Kneser Theorem

- Case I: If $|P(w)| + |N(w)| \leq 2k - 2$, then set

$$\lambda(w) := \begin{cases} |P(w)| + |N(w)| + 1 & if\ |P(w)| \geq |N(w)| \\ -(|P(w)| + |N(w)| + 1) & if\ |P(w)| < |N(w)|. \end{cases}$$

- Case II: If $|P(w)| + |N(w)| \geq 2k - 1$, then set

$$\lambda(w) := \begin{cases} t_P & if\ |P(w)| \geq |N(w)| \\ -t_N & if\ |P(w)| < |N(w)|. \end{cases}$$

where t_P is the largest positive integer such that there exists a k-subset $A \subseteq P(w)$ where $c(A) = t_P$. Similarly, define t_N!

Lovász-Kneser Theorem

- Case I: If $|P(w)| + |N(w)| \leq 2k-2$, then set

$$\lambda(w) := \begin{cases} |P(w)| + |N(w)| + 1 & if\ |P(w)| \geq |N(w)| \\ -(|P(w)| + |N(w)| + 1) & if\ |P(w)| < |N(w)|. \end{cases}$$

- Case II: If $|P(w)| + |N(w)| \geq 2k-1$, then set

$$\lambda(w) := \begin{cases} t_P & if\ |P(w)| \geq |N(w)| \\ -t_N & if\ |P(w)| < |N(w)|. \end{cases}$$

where t_P is the largest positive integer such that there exists a k-subset $A \subseteq P(w)$ where $c(A) = t_P$. Similarly, define t_N!

Ky Fan's Theorem for S^n

- (Ky Fan's Theorem) Let T be a triangulation of S^n that is antipodally symmetric. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling of the vertices of T in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every vertex $v \in T$.
- There is no antipodal edge, i.e., for any 1-simplex in T, the numbers assigned to its two vertices have sum distinct from zero.
- Then there exists an n-simplex in T whose vertices receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n + 1$.

Ky Fan's Theorem for S^n

- (Ky Fan's Theorem) Let T be a triangulation of S^n that is antipodally symmetric. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \dots, -m, +m\}$ be a labeling of the vertices of T in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every vertex $v \in T$.
- There is no antipodal edge, i.e., for any 1-simplex in T, the numbers assigned to its two vertices have sum distinct from zero.
- Then there exists an n-simplex in T whose vertices receive the numbers $-a_1, a_2, \dots, (-1)^{n+1}a_{n+1}$, where $1 \leq a_1 < a_2 < \dots < a_{n+1} \leq m$.
- In particular, $m \geq n + 1$.

Ky Fan's Theorem for S^n

- (Ky Fan's Theorem) Let T be a triangulation of S^n that is antipodally symmetric. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling of the vertices of T in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every vertex $v \in T$.
- There is no antipodal edge, i.e., for any 1-simplex in T, the numbers assigned to its two vertices have sum distinct from zero.
- Then there exists an n-simplex in T whose vertices receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n + 1$.

Ky Fan's Theorem for S^n

- (Ky Fan's Theorem) Let T be a triangulation of S^n that is antipodally symmetric. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling of the vertices of T in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every vertex $v \in T$.
- There is no antipodal edge, i.e., for any 1-simplex in T, the numbers assigned to its two vertices have sum distinct from zero.
- Then there exists an n-simplex in T whose vertices receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n + 1$.

Ky Fan's Theorem for S^n

- (Ky Fan's Theorem) Let T be a triangulation of S^n that is antipodally symmetric. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling of the vertices of T in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every vertex $v \in T$.
- There is no antipodal edge, i.e., for any 1-simplex in T, the numbers assigned to its two vertices have sum distinct from zero.
- Then there exists an n-simplex in T whose vertices receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n + 1$.

Ky Fan's Theorem for B^n

- (Ky Fan's Theorem) Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Let $\lambda : V(T) \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling of the vertices of T in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every vertex $v \in T$ lying on the boundary.
- There is no antipodal edge, i.e., for any 1-simplex in T, the numbers assigned to its two vertices have sum distinct from zero.
- Then there exists an n-simplex in T whose vertices receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n + 1$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$ (mod 2).
- $\sum_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum_{a_1} \sum_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum_{a_1} \sum_{0<a_2 \neq a_1} \alpha(a_1, -a_2)$ (mod 2).

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] \pmod 2$.
- $\sum_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum_{a_1} \sum_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum_{a_1} \sum_{0<a_2 \neq a_1} \alpha(a_1, -a_2) \pmod 2$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] \pmod 2$.
- $\sum_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum_{a_1} \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum_{a_1} \sum_{0<a_2\neq a_1} \alpha(a_1, -a_2) \pmod 2$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] \pmod 2$.
- $\sum_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum_{a_1} \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum_{a_1} \sum_{0<a_2\neq a_1} \alpha(a_1, -a_2) \pmod 2$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] \pmod 2$.
- $\sum_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum_{a_1} \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum_{a_1} \sum_{0<a_2\neq a_1} \alpha(a_1, -a_2) \pmod 2$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] \pmod 2$.
- $\sum_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum_{a_1} \sum_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum_{a_1} \sum_{0<a_2\neq a_1} \alpha(a_1, -a_2) \pmod 2$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum\limits_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum\limits_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] \pmod 2$.
- $\sum\limits_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum\limits_{a_1} \sum\limits_{0<a_2\neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum\limits_{a_1} \sum\limits_{0<a_2\neq a_1} \alpha(a_1, -a_2) \pmod 2$.

Sketch of Proof for n=1!

- Define $\gamma(j) := 1$ whenever $\lambda(P) = j$; otherwise set $\gamma(j) := 0$.
- Set $\delta(j) := |\{v : v \in T \setminus \{P, Q\}, \lambda(v) = j\}|$.
- $\alpha(a, b) :=$ the total number of those 1-simplices in T, whose 2 vertices receive the numbers a and b. By double counting:
- $\gamma(a_1) + \gamma(-a_1) + 2\delta(a_1) = 2\alpha(a_1, a_1) + \sum\limits_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$.
- $\gamma(a_1) + \gamma(-a_1) = \sum\limits_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)]$ (mod 2).
- $\sum\limits_{a_1} [\gamma(a_1) + \gamma(-a_1)] = 1$
- $\sum\limits_{a_1} \sum\limits_{0<a_2 \neq a_1} [\alpha(a_1, a_2) + \alpha(a_1, -a_2)] = \sum\limits_{a_1} \sum\limits_{0<a_2 \neq a_1} \alpha(a_1, -a_2)$ (mod 2).

Combinatorial Ky Fan's Theorem

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Ky Fan's Theorem) Let $\lambda : V_n \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$.
- For any $u, v \in V_n$ such that $u \leq v$, we have $\lambda(u) + \lambda(v) \neq 0$.
- Then there exists a $n+1$-chain in V_n whose memberes receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1} a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n+1$.

Combinatorial Ky Fan's Theorem

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Ky Fan's Theorem) Let $\lambda : V_n \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$.
- For any $u, v \in V_n$ such that $u \leq v$, we have $\lambda(u) + \lambda(v) \neq 0$.
- Then there exists a $n+1$-chain in V_n whose memberes receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1} a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n+1$.

Combinatorial Ky Fan's Theorem

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Ky Fan's Theorem) Let $\lambda : V_n \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$.
- For any $u, v \in V_n$ such that $u \leq v$, we have $\lambda(u) + \lambda(v) \neq 0$.
- Then there exists a $n+1$-chain in V_n whose memberes receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n+1$.

Combinatorial Ky Fan's Theorem

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Ky Fan's Theorem) Let $\lambda : V_n \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$.
- For any $u, v \in V_n$ such that $u \leq v$, we have $\lambda(u) + \lambda(v) \neq 0$.
- Then there exists a $n+1$-chain in V_n whose memberes receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1} a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n+1$.

Combinatorial Ky Fan's Theorem

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Ky Fan's Theorem) Let $\lambda : V_n \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$.
- For any $u, v \in V_n$ such that $u \leq v$, we have $\lambda(u) + \lambda(v) \neq 0$.
- Then there exists a $n+1$-chain in V_n whose memberes receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n+1$.

Combinatorial Ky Fan's Theorem

- Set $V_n := \{-1, 0, +1\}^n$.
- Consider a partial ordering $\leq$ on V_n as follows:
- For any $u, v \in V_n$ we have $u \leq v$ if $u_i \leq v_i$ for any $1 \leq i \leq n$ where $0 \leq -1$ and $0 \leq +1$.
- (Combinatorial Ky Fan's Theorem) Let $\lambda : V_n \longrightarrow \{-1, +1, -2, +2, \ldots, -m, +m\}$ be a labeling in such a way that the following conditions are satisfied:
- $\lambda(-v) = -\lambda(v)$ for every $v \in V_n$ where $v \neq 0$.
- For any $u, v \in V_n$ such that $u \leq v$, we have $\lambda(u) + \lambda(v) \neq 0$.
- Then there exists a $n+1$-chain in V_n whose memberes receive the numbers $-a_1, a_2, \ldots, (-1)^{n+1}a_{n+1}$ or $a_1, -a_2, \ldots, (-1)^n a_{n+1}$, where $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq m$.
- In particular, $m \geq n+1$.

References

Ky Fan, A generalization of Tucker's combinatorial lemma with topological applications.
Ann. of Math. (2), 56:431–437, 1952.

Jiří Matoušek, *Using the Borsuk-Ulam theorem*, Universitext. Springer-Verlag, Berlin, 2003.
Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.

Jiří Matoušek, A combinatorial proof of Kneser's conjecture.
Combinatorica, 24(1):163–170, 2004.

A. W. Tucker, Some topological properties of disk and sphere.
In *Proc. First Canadian Math. Congress, Montreal, 1945*, pages 285–309. University of Toronto Press, Toronto, 1946.

Thank You!