Applications of Borsuk-Ulam Theorem

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Contents

Level of Rings

- Let R be a ring. The level $s(R)$ of R is the smallest n such that -1 can be written as the sum of n squares in R, that is, $-1 = r_1^2 + \cdots + r_n^2$ for some $r_1, \ldots, r_n \in R$.
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- Z.D. Dai, T.Y. Lam, and C.K. Peng have given an affirmative answer to the aforementioned question.
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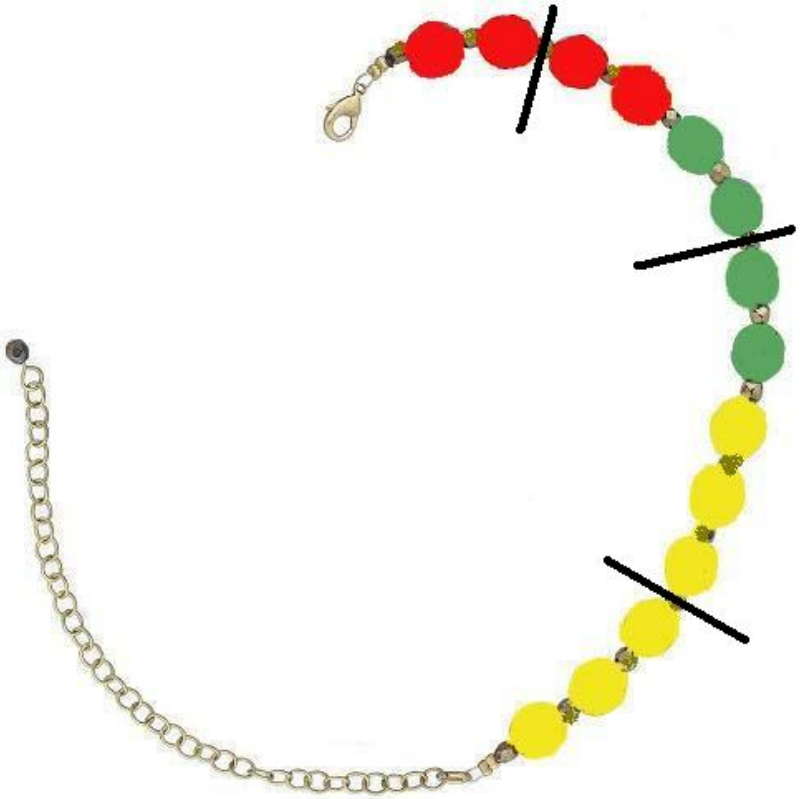

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- **Interval coloring**. Let $I = [0, 1]$ be the unit interval. Suppose that every point of I has a color i, $1 \leq i \leq k$, such that for each i the set of points colored i is measurable. Call such a coloring of I an interval coloring.
- **The Continuous Problem**. Given an interval coloring, a bisection of size r is a sequence of numbers $0 = y_o < y_1 < \cdots < y_r < y_{r+1} = 1$ such that $\cup\{[y_{i-1}, y_i] : i = 0 \bmod 2\}$ captures precisely half the measure of each color. *Is there any bisection of size at most k for given interval coloring with k colors?*

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Proof of The Continuous Problem

- Given an interval k-coloring of $[0,1]$, define a function $f : S^k \to \mathbb{R}^k$ as follows.
- If $x \in S^k$, then set $\alpha(x) := (z_0, z_1, \ldots, z_k)$ where $z_0 = 0$ and $z_j = \sum_{i=1}^{j} x_i^2$ for $j \geq 1$.
- For $1 \leq j \leq k$, define $f_j(x) = \sum_{i=1}^{k+1} sign(x_i) m_j(i)$, where $m_j(i)$ is the measure of the jth color in the segment $[z_{i-1}, z_i]$.
- Set $f(x) = (f_1(x), \ldots, f_k(x))$. Clearly $f : S^k \to \mathbb{R}^k$ and $f(-x) = -f(x)$. Hence, there exists $x \in S^k$ such that $f(x) = 0$.
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Hobby and Rice Theorem

- **Hobby and Rice Theorem**. Let $g_1, \ldots, g_k : [0,1] \to \mathbb{R}$ be k continuously-integrable functions. Then there exist

$$0 = z_o \leq z_1 \leq \cdots \leq z_k \leq z_{k+1} = 1$$

and

$$\delta_1, \ldots, \delta_{k+1} \in \{-1, +1\}$$

such that $\sum_{i=1}^{k+1} \delta_i \int_{z_{i-1}}^{z_i} g_j = 0$ for all $1 \leq j \leq k$.

The Continuous Problem

- **Necklace Splitting Theorem**. If p thieves want to split a necklace with k kinds of beads such that each of them get $\lfloor \frac{a_i}{p} \rfloor$ or $\lceil \frac{a_i}{p} \rceil$ beads of i^{th}th kind, then they can do so using at most $(p-1)k$ cuts.
- General Necklace Splitting Conjecture. If p thieves want to split a necklace with k kinds of beads such that the jth thief gets $a_i^{(j)}$ of the ith kind where $a_i^{(j)} = \lfloor \frac{a_i}{p} \rfloor$ or $a_i^{(j)} = \lceil \frac{a_i}{p} \rceil$ and $\sum_j a_i^{(j)} = a_i$, then they can do so using at most $(p-1)k$ cuts.

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Ham Sandwich Theorem

- Ham sandwich theorem for measures. Let $\mu_1, \mu_2, \ldots, \mu_d$ be finite Borel measures on R^d such that every hyperplane has measure 0 for each of the μ_i (in the sequel, we refer to such measures as mass distributions). Then there exists a hyperplane h such that $\mu_i(h^+) = \frac{1}{2}\mu_i(R^d)$ for $i = 1, 2, \ldots, d$ where h^+ denotes one of the half-spaces defined by h.

Sketch of Proof

- Let $\mathbf{u} = (u_0, u_1, \ldots, u_d)$ be a point of the sphere S^d.
- If at least one of the components $u_1, u_2, ..., u_d$ is nonzero, we assign to the point $\mathbf{u}$ the half-space

$$h^+(\mathbf{u}) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d | u_1 x_1 + \ldots + u_d x_d \leq u_0\}.$$

- Define $f : S^d \to \mathbb{R}^d$ by $f_i(\mathbf{u}) = \mu_i(h^+(\mathbf{u}))$.
- One check that $f : S^d \to \mathbb{R}^d$ is continuous, then $f(\mathbf{u}) = f(-\mathbf{u})$ for some $\mathbf{u} \in \mathbb{R}^d$.
- In other words, $\mu_i(h^+(\mathbf{u})) = \frac{1}{2}\mu_\mathbf{i}(\mathbf{R^d})$

Sketch of Proof

- Let $\mathbf{u} = (u_0, u_1, \ldots, u_d)$ be a point of the sphere S^d.
- If at least one of the components $u_1, u_2, ..., u_d$ is nonzero, we assign to the point $\mathbf{u}$ the half-space

$$h^+(\mathbf{u}) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d | u_1 x_1 + \ldots + u_d x_d \leq u_0\}.$$

- Define $f : S^d \to \mathbb{R}^d$ by $f_i(\mathbf{u}) = \mu_i(h^+(\mathbf{u}))$.
- One check that $f : S^d \to \mathbb{R}^d$ is continuous, then $f(\mathbf{u}) = f(-\mathbf{u})$ for some $\mathbf{u} \in \mathbb{R}^d$.
- In other words, $\mu_i(h^+(\mathbf{u})) = \frac{1}{2}\mu_\mathbf{i}(\mathbf{R^d})$

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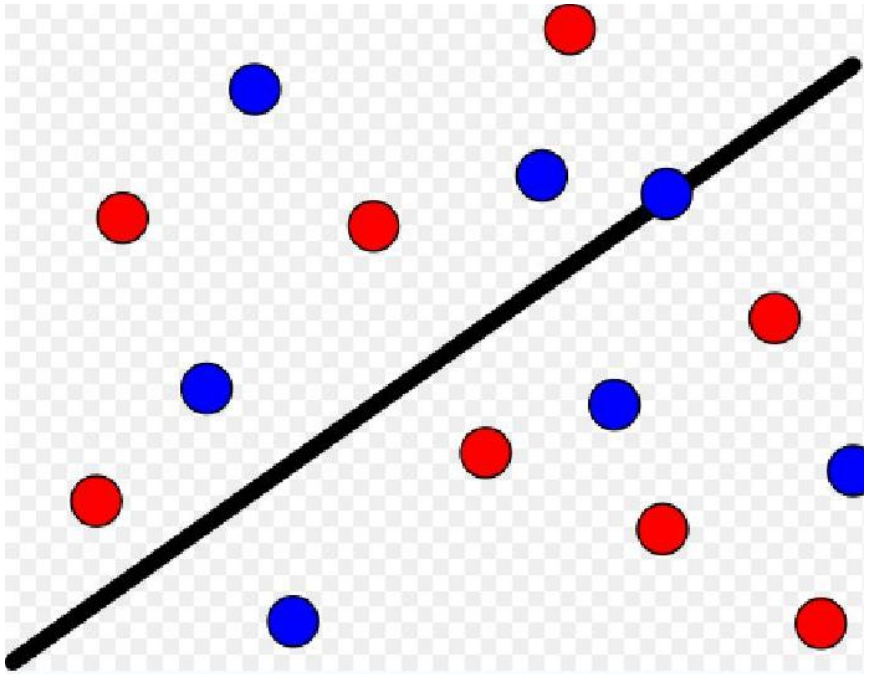
Ham Sandwich Theorem

- Ham sandwich theorem for point sets. Let $A_1, A_2, \ldots, A_d \subset R^d$ be finite point sets. Then there exists a hyperplane h that simultaneously bisects $A_1, A_2, \ldots, A_d \subset R^d$.



Team-Splitting

- (Team-Splitting) Given a territory and such a collection of $2n$ explorers(e.g. two zoologist, two botanists, two archaeologist etc), there exists a way to divide the territory and the people into two teams of n explorers(one of each type)such that each explorer is satisfied with his/her territory.

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Some applications

- (Consensus-Halving) Consider an object A, and n people whose preferences on A are modeled by continuous measure $\mu_1, \ldots, \mu_n$. Using at most n cuts by parallel planes, A may be divided into two portions A^+ and A^- such that each of n people thinks that A^+ and A^- are exactly equal, i.e., $\mu_i(A^+) = \mu_i(A^-)$.

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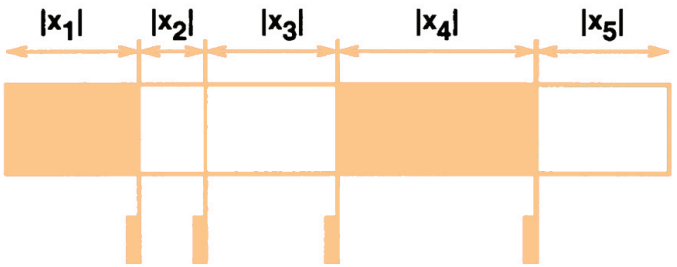

Figure: A cut-set represented by $(-0.2, +0.1, +0.2, -0.3, +0.2)$. The portion A_+ is the union of the white pieces; A_- is the union of the shaded pieces.

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Thank You!