Applications of Borsuk-Ulam Theorem

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Level of Rings
Necklace Theorem
Ham Sandwich Theorem
Team-Splitting
Consensus-Halving Theorem
**Level of Rings**

- Let $R$ be a ring. The level $s(R)$ of $R$ is the smallest $n$ such that $-1$ can be written as the sum of $n$ squares in $R$, that is, $-1 = r_1^2 + \cdots + r_n^2$ for some $r_1, \ldots, r_n \in R$.
- By a theorem of Pfister, the level of every field is either $\infty$ or a power of 2.
- *Is there a ring of level $n$ for every $n$?*
- Z.D. Dai, T.Y. Lam, and C.K. Peng have given an affirmative answer to the aforementioned question.
- For a given $\mathbb{Z}_2$-space $X$, $\text{ind}_{\mathbb{Z}_2}(X) = s(R_X) - 1$, where $R_X$ is the ring of all $\mathbb{Z}_2$-maps $X \to \mathbb{C}$, with the $\mathbb{Z}_2$-action on $\mathbb{C}$ being the complex conjugation.
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**Applications of Borsuk-Ulam Theorem**

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Set \( R_n := \mathbb{R}[t_1, \ldots, t_n] / <1 + t_1^2 + t_2^2 + \cdots + t_n^2> \). Clearly, \( s(R_n) \leq n \).

Let \( s(R_n) = m < n \). There are \( f_0, f_1, \ldots, f_m \in \mathbb{R}[t_1, \ldots, t_n] \) such that \( p := f_1^2 + f_2^2 + \cdots + f_m^2 + f_0(1 + t_1^2 + t_2^2 + \cdots + t_n^2) \) is identically \(-1\).

For a point \( x \in S^{n-1} \), define \( q_j(x) \) as the imaginary part of \( f_j(w) \), where \( w \) is the complex vector \((ix_1, ix_2, \ldots, ix_n)\).

\( q_j(-x) = -q_j(x) \), hence, \( q : x \to (q_1(x), q_2(x), \ldots, q_m(x)) \) is an antipodal map \( S^{n-1} \to \mathbb{R}^m \).

We claim that \( q(x) \neq 0 \) for every \( x \in S^{n-1} \), which contradicts the Borsuk-Ulam theorem.

Indeed, we have \( p := f_1^2 + f_2^2 + \cdots + f_m^2 = -1 \). But, this equality could not hold if the imaginary parts of the \( f_j(w) \) were all \( 0 \).
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Necklace Theorem

- **Interval coloring.** Let $I = [0, 1]$ be the unit interval. Suppose that every point of $I$ has a color $i$, $1 \leq i \leq k$, such that for each $i$ the set of points colored $i$ is measurable. Call such a coloring of $I$ an interval coloring.

- **The Continuous Problem.** Given an interval coloring, a bisection of size $r$ is a sequence of numbers $0 = y_0 < y_1 < \cdots < y_r < y_{r+1} = 1$ such that $\bigcup \{ [y_{i-1}, y_i] : i = 0 \mod 2 \}$ captures precisely half the measure of each color. Is there any bisection of size at most $k$ for given interval coloring with $k$ colors?
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Proof of The Continuous Problem

- Given an interval $k$-coloring of $[0, 1]$, define a function $f : S^k \to \mathbb{R}^k$ as follows.
  - If $x \in S^k$, then set $\alpha(x) := (z_0, z_1, \ldots, z_k)$ where $z_0 = 0$ and $z_j = \sum_{i=1}^{j} x_i^2$ for $j \geq 1$.
  - For $1 \leq j \leq k$, define $f_j(x) = \sum_{i=1}^{k+1} \text{sign}(x_i)m_j(i)$, where $m_j(i)$ is the measure of the $j$th color in the segment $[z_{i-1}, z_i]$.

- Set $f(x) = (f_1(x), \ldots, f_k(x))$. Clearly $f : S^k \to \mathbb{R}^k$ and $f(-x) = -f(x)$. Hence, there exists $x \in S^k$ such that $f(x) = 0$.

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Hobby and Rice Theorem. Let $g_1, \ldots, g_k : [0, 1] \to \mathbb{R}$ be $k$ continuously-integrable functions. Then there exist

$$0 = z_0 \leq z_1 \leq \cdots \leq z_k \leq z_{k+1} = 1$$

and

$$\delta_1, \ldots, \delta_{k+1} \in \{-1, +1\}$$

such that

$$\sum_{i=1}^{k+1} \delta_i \int_{z_{i-1}}^{z_i} g_j = 0 \quad \text{for all } 1 \leq j \leq k.$$
The Continuous Problem

- **Necklace Splitting Theorem.** If \( p \) thieves want to split a necklace with \( k \) kinds of beads such that each of them get \( \lfloor \frac{a_i}{p} \rfloor \) or \( \lceil \frac{a_i}{p} \rceil \) beads of \( i^{th} \) kind, then they can do so using at most \((p - 1)k\) cuts.

- **General Necklace Splitting Conjecture.** If \( p \) thieves want to split a necklace with \( k \) kinds of beads such that the \( j \)th thief gets \( a_i^{(j)} \) of the \( i \)th kind where \( a_i^{(j)} = \lfloor \frac{a_i}{p} \rfloor \) or \( a_i^{(j)} = \lceil \frac{a_i}{p} \rceil \) and \( \sum_j a_i^{(j)} = a_i \), then they can do so using at most \((p - 1)k\) cuts.
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The Continuous Problem

- **Necklace Splitting Theorem.** If \( p \) thieves want to split a necklace with \( k \) kinds of beads such that each of them get \( \left\lfloor \frac{a_i}{p} \right\rfloor \) or \( \left\lceil \frac{a_i}{p} \right\rceil \) beads of \( i^{th} \)th kind, then they can do so using at most \((p - 1)k\) cuts.

- **General Necklace Splitting Conjecture.** If \( p \) thieves want to split a necklace with \( k \) kinds of beads such that the \( j \)th thief gets \( a_i^{(j)} \) of the \( i \)th kind where \( a_i^{(j)} = \left\lfloor \frac{a_i}{p} \right\rfloor \) or \( a_i^{(j)} = \left\lceil \frac{a_i}{p} \right\rceil \) and \( \sum_j a_i^{(j)} = a_i \), then they can do so using at most \((p - 1)k\) cuts.
Ham Sandwich Theorem

- Ham sandwich theorem for measures. Let $\mu_1, \mu_2, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each of the $\mu_i$ (in the sequel, we refer to such measures as mass distributions). Then there exists a hyperplane $h$ such that $\mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^d)$ for $i = 1, 2, \ldots, d$ where $h^+$ denotes one of the half-spaces defined by $h$. 
Sketch of Proof

Let $u = (u_0, u_1, \ldots, u_d)$ be a point of the sphere $S^d$.

If at least one of the components $u_1, u_2, \ldots, u_d$ is nonzero, we assign to the point $u$ the half-space

$$h^+(u) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d | u_1 x_1 + \ldots + u_d x_d \leq u_0\}.$$

Define $f : S^d \rightarrow \mathbb{R}^d$ by $f_i(u) = \mu_i(h^+(u))$.

One check that $f : S^d \rightarrow \mathbb{R}^d$ is continuous, then $f(u) = f(-u)$ for some $u \in \mathbb{R}^d$.

In other words, $\mu_i(h^+(u)) = \frac{1}{2} \mu_i(\mathbb{R}^d)$.
Sketch of Proof

- Let $\mathbf{u} = (u_0, u_1, \ldots, u_d)$ be a point of the sphere $S^d$.
- If at least one of the components $u_1, u_2, \ldots, u_d$ is nonzero, we assign to the point $\mathbf{u}$ the half-space
  \[ h^+(\mathbf{u}) := \{ (x_1, \ldots, x_d) \in \mathbb{R}^d | u_1 x_1 + \ldots + u_d x_d \leq u_0 \} \].
- Define $f : S^d \rightarrow \mathbb{R}^d$ by $f_i(\mathbf{u}) = \mu_i(h^+(\mathbf{u}))$.
- One check that $f : S^d \rightarrow \mathbb{R}^d$ is continuous, then
  $f(\mathbf{u}) = f(-\mathbf{u})$ for some $\mathbf{u} \in \mathbb{R}^d$.
- In other words, $\mu_i(h^+(\mathbf{u})) = \frac{1}{2} \mu_i(\mathbb{R}^d)$.
Sketch of Proof

- Let \( \mathbf{u} = (u_0, u_1, \ldots, u_d) \) be a point of the sphere \( S^d \).
- If at least one of the components \( u_1, u_2, \ldots, u_d \) is nonzero, we assign to the point \( \mathbf{u} \) the half-space
  \[
  h^+(\mathbf{u}) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d | u_1 x_1 + \ldots + u_d x_d \leq u_0 \}.
  \]
- Define \( f : S^d \to \mathbb{R}^d \) by \( f_i(\mathbf{u}) = \mu_i(h^+(\mathbf{u})) \).
- One check that \( f : S^d \to \mathbb{R}^d \) is continuous, then
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  f(\mathbf{u}) = f(-\mathbf{u}) \text{ for some } \mathbf{u} \in \mathbb{R}^d.
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- In other words, \( \mu_i(h^+(\mathbf{u})) = \frac{1}{2} \mu_i(\mathbb{R}^d) \).
Applications of Borsuk-Ulam Theorem

Ham Sandwich Theorem

**Sketch of Proof**

- Let \( u = (u_0, u_1, \ldots, u_d) \) be a point of the sphere \( S^d \).
- If at least one of the components \( u_1, u_2, \ldots, u_d \) is nonzero, we assign to the point \( u \) the half-space
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- Define \( f : S^d \to \mathbb{R}^d \) by \( f_i(u) = \mu_i(h^+(u)) \).
- One check that \( f : S^d \to \mathbb{R}^d \) is continuous, then \( f(u) = f(-u) \) for some \( u \in \mathbb{R}^d \).
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Sketch of Proof

- Let \( u = (u_0, u_1, \ldots, u_d) \) be a point of the sphere \( S^d \).
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h^+(u) := \{(x_1, \ldots, x_d) \in \mathbb{R}^d | u_1x_1 + \ldots + u_dx_d \leq u_0 \}.
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- In other words, \( \mu_i(h^+(u)) = \frac{1}{2} \mu_i(\mathbb{R}^d) \).
**Ham Sandwich Theorem**

- **Ham sandwich theorem for point sets.** Let $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane $h$ that simultaneously bisects $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$. 

![Diagram of Ham Sandwich Theorem](image-url)
Team-Splitting

Given a territory and such a collection of $2n$ explorers (e.g. two zoologists, two botanists, two archaeologists etc), there exists a way to divide the territory and the people into two teams of $n$ explorers (one of each type) such that each explorer is satisfied with his/her territory.
Team-Splitting

▶ (Team-Splitting) Given a territory and such a collection of $2n$ explorers (e.g. two zoologists, two botanists, two archaeologists etc), there exists a way to divide the territory and the people into two teams of $n$ explorers (one of each type) such that each explorer is satisfied with his/her territory.
Some applications

- (Consensus-Halving) Consider an object $A$, and $n$ people whose preferences on $A$ are modeled by continuous measure $\mu_1, \ldots, \mu_n$. Using at most $n$ cuts by parallel planes, $A$ may be divided into two portions $A^+$ and $A^-$ such that each of $n$ people thinks that $A^+$ and $A^-$ are exactly equal, i.e., $\mu_i(A^+) = \mu_i(A^-)$.
Some applications

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Some applications

- (Consensus-Halving) Consider an object $A$, and $n$ people whose preferences on $A$ are modeled by continuous measure $\mu_1, \ldots, \mu_n$. Using at most $n$ cuts by parallel planes, $A$ may be divided into two portions $A^+$ and $A^-$ such that each of $n$ people thinks that $A^+$ and $A^-$ are exactly equal, i.e., $\mu_i(A^+) = \mu_i(A^-)$.

![Diagram showing a cut-set represented by $(-0.2, +0.1, +0.2, -0.3, +0.2)$. The portion $A^+$ is the union of the white pieces; $A^-$ is the union of the shaded pieces.](image-url)
References


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Thank You!